Spin-Space Entanglement Transfer and Quantum Statistics

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Abstract

Both the topics of entanglement and particle statistics have aroused enormous research interest since the advent of quantum mechanics. Using two pairs of entangled particles we show that indistinguishability enforces a transfer of entanglement from the internal to the spatial degrees of freedom. Moreover, sub-ensembles selected by measurements on internal degrees of freedom will in general have different amounts of entanglement between the paths depending on the statistics (either fermionic or bosonic) of the particles involved. This establishes a difference between fermions and bosons in terms of their information processing power.

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Since the advent of quantum mechanics, entanglement has been identified as one of its most peculiar features [1–3]. This "excess correlation" has recently become an important resource in quantum information processing [4]. Entanglement is believed to be at the root of the speed-up of quantum computers over their classical counterparts [5], and it also leads to an unconditionally secure quantum cryptographic key exchange [6]. Another fundamental aspect of quantum physics, somewhat neglected in the field of quantum information (although see [7]), is the distinction between two different types of particles, fermions and bosons, manifested through particle statistics. There are at first sight two seemingly "conflicting" views regarding the role of indistinguishability and particle statistics in quantum information processing. On the one hand, these two notions appear to combine to offer "natural" entanglement through forcing the use of symmetrised and anti-symmetrised states (for bosons and fermions respectively), and as we mentioned before, entanglement is generally an advantage for quantum information processing (although see [8]). On the other hand, indistinguishability prevents us from addressing the particles separately which seems to be disadvantage in information processing. In this letter we analyze the role of indistinguishability and particle statistics in a simple information processing scenario.

Consider the following situation. Suppose that we have two pairs of qubits (quantum two-level systems), each pair maximally entangled in some internal degree of freedom. If the particles carrying the qubits are of the same type – say bosons – but distinguishable as a result of spatial separation, then we have two units of entanglement (e-bits) in total. If we now consider bringing the particles close together, then we will have a fully indistinguishable situation, and the number of internal degrees of freedom is reduced from 16 to 5 (from 2^4 to $4+1$), due to symmetrization. We cannot now exceed $1/2 \log 5$ e-bits of entanglement which is less then the previous 2 e-bits. Assuming that the operation bringing the particles close together is local unitary (which can be the case as shown below), then the total entanglement should be conserved, which creates an apparent paradox as some entanglement seems to have been lost due to the indistinguishability. The solution to this problem is that entanglement is transferred to the spatial degrees of freedom so that the total entanglement (spatial and internal degrees of freedom together) is, of course, conserved. The fascinating implication is that the transfer of entanglement is imposed by particle indistinguishability and does not involve any controlled operation between the internal and external degrees of freedom (i.e. spin-path interaction), in contrast with the standard entanglement swapping scheme [9]. Moreover, the amount of entanglement transferred will, in general, depend on the nature of the particles involved, i.e. fermions and bosons will give different results. Turning this around, we can use the entanglement information processing capacity to determine the difference between the two different fundamental types of particles in nature. One might expect that due to the Pauli exclusion principle, more entanglement will be transferred to space in the case of fermions.

Now we turn to describing the exact details of our thought experiment. Imagine the following setup, described in Fig. 1. We have two pairs of identical particles, each pair being maximally entangled in some internal degree of freedom, e.g. the spin, or polarization. In our case, we consider systems with spin one-half, or isomorphic to it. We assume that our setup is symmetrical both horizontally and vertically, where the dotted lines in Fig. 1 show the axis of symmetry. We have to ensure that particles arrive at the beam splitter at the same time. The initial entanglement is between sides 1 and 2. In each pair, the particles fly apart and meet a particle from the other pair at a beam splitter. The paths on the left hand side are labeled A and C respectively before and after the beam splitter. Similarly, paths on the right hand side are labeled B and D.

The output states of this setup represent particles in paths $C1$, $D1$, $C2$ and $D2$ with a particular spin state (we note, for instance, that we can have two particles in $C1$ and none in $D1$). Now we show that, although the initial entanglement is only in the internal degrees of freedom, in the final state some of the entanglement has been transferred to the paths. Moreover, this entanglement is different depending on whether the particles are fermions or bosons. We will refer to this effect as the *Spin-Space Entanglement Transfer* only by local actions.

In order to calculate what happens in the above setup, we write our initial state in the second quantization formalism:

$$\frac{1}{\sqrt{2}}(a^\dagger_{A1\uparrow}a^\dagger_{A2\downarrow} \pm a^\dagger_{A1\downarrow}a^\dagger_{A2\uparrow})\frac{1}{\sqrt{2}}(a^\dagger_{B1\uparrow}a^\dagger_{B2\downarrow} \pm a^\dagger_{B1\downarrow}a^\dagger_{B2\uparrow})|0\rangle \tag{1}$$

where $|0\rangle$ is the vacuum state and, for instance, $a^\dagger_{A1\uparrow}$ is a creation operator describing a particle in path A1 and with spin up. The positive and negative signs in the above equation are necessary in order to take into account all the possible initial states (the singlet and the entangled triplet of spin). We restrict our attention to analyzing one mode per particle only, but our results can be generalized to any number of modes. Due to the symmetry of the problem we only need to analyze two cases: when the two signs in equation (1) are the same, the $(+,+)$ case, and when they are different, the $(+,-)$ case.

The operation of the beam splitter is described by any unitary transformation in $U(2)$ [10]. However, since the overall phase factor has no relevance for entanglement, we can without any loss of generality consider a transformation in $SU(2)$:

$$U = \begin{bmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{bmatrix} \tag{2}$$

where $|\alpha|^2 + |\beta|^2 = 1$. Since we consider entanglement only between sides 1 and 2, the beam splitters in fact perform local unitary operations. Hence they cannot change the total entanglement present initially. Also, they only affect the spatial degrees of freedom and are not intrinsically dependent on spin (or polarization). Therefore they are incapable of swapping entanglement from spin (polarization) to space by performing a controlled not operation in the usual fashion [9]. Although the transformation law will be the same for fermions and bosons, they obey different statistics which is why there will be an observable difference in their behaviour in our experiment. For fermions we have the following anti-commutation relation:

$$[a^\dagger_i, a^\dagger_j]_+ = 0, \tag{3}$$

while for bosons we have the commutation relation:

$$[a_i^\dagger, a_j^\dagger]_- = 0, \tag{4}$$

where i and j are sets of labels. Figs. 2 to 5 diagramatically present the output states for both fermions and bosons. For instance, the first diagram in Fig. 4 represents the following term:

$$(|\alpha|^2 + |\beta|^2)^2 \, (a_{C1\uparrow}^\dagger a_{D1\uparrow}^\dagger a_{C2\downarrow}^\dagger a_{D2\downarrow}^\dagger)|0\rangle \tag{5}$$

As the total spin (polarization) S on sides 1 and 2 can either be 0 or 1, we can divide the total output wave function into these two components.

S = 0 component: there is no difference between fermions and bosons (bearing in mind that the corresponding operators obey different commutation relations). However, there is a difference between the $(+,+)$ case, where we have all possible output terms (see Fig. 2), and the $(+,-)$ case, where some terms never appear (see Fig. 3).

S = 1 component: there is a difference between the output states for fermions (see Fig. 4) and bosons (see Fig. 5). For both types of particles, the $(+,-)$ case will only introduce a phase difference in some terms.

As a consequence of applying only local unitary operations, the total output wave function should have also two e-bits of entanglement. For clarity, let us consider for the rest of the paper the particular case of 50/50 beam splitters ($\alpha = \frac{1}{\sqrt{2}}, \beta = \frac{-i}{\sqrt{2}}$). To illustrate the spin-space entanglement transfer effect, we look at the $(+,+)$ case for fermions (Figs. 2 and 4). Here, it is clear that the $S = 1$ terms give one e-bit of entanglement, solely in the internal degrees of freedom, as the path states are identical. The $S = 0$ case gives the other e-bit of entanglement, but this time involving both the internal and external degrees of freedom. Thus we have spin-space entanglement transfer, without any controlled operation between spin and space.

We now show how we can extract space-only entanglement from the total wave function by doing particular measurements on the internal degrees of freedom without revealing any knowledge about the external ones. For example, we can measure all four spins in the x

direction. Suppose all the spins are found to be in the $+x$ state. For bosons, the entire wave function is then projected onto:

$$\frac{1}{\sqrt{2}}\left[|L\rangle_1\left(\frac{2}{\sqrt{5}}|L\rangle_2+\frac{1}{\sqrt{5}}|R\rangle_2\right)+|R\rangle_1\left(\frac{2}{\sqrt{5}}|R\rangle_2+\frac{1}{\sqrt{5}}|L\rangle_2\right)\right] \tag{6}$$

where $|L\rangle_{1,2}$ means left bunching of the particles, respectively for sides 1 and 2, and $|R\rangle_{1,2}$ right bunching. The fermionic counterpart of the above state is:

$$\frac{1}{\sqrt{5}}\left[-\frac{1}{\sqrt{2}}\left(|L\rangle_1+|R\rangle_1\right)\frac{1}{\sqrt{2}}\left(|L\rangle_2+|R\rangle_2\right)\right]+\frac{2}{\sqrt{5}}|A\rangle_1|A\rangle_2 \tag{7}$$

where $|A\rangle_{1,2}$ represents anti-bunching. The degree of path entanglement, given by the von Neumann entropy, is thus 0.47 for bosons and 0.72 for fermions (even the other three outcomes of the spin measurement along x give the same results). This confirms our initial expectation that more entanglement will be transferred to space due to the Pauli pressure. We can obtain an even greater difference in path entanglement between these two types of particles if we first select out the $S = 1$ sub-ensemble, but in this case, bosons will now have 1 e-bit of entanglement, whereas fermions will have no space entanglement (our initial intuition no longer applies, as in this case we are not looking at the whole state). Measuring this degree of entanglement could thus be an operational way of distinguishing between bosons and fermions. This also establishes a connection between two fundamental notions of quantum physics: entanglement and particle statistics. We intend to present a more detailed and systematic analysis of this setup in a subsequent longer work.

Our analysis suggests further investigation of the consequences and applications of particle statistics in quantum information processing. Other types of statistics (e.g. anyons) can similarly be addressed within our framework. Recent experiments such as [11,12] suggest that it would be possible to test our results in the near future.

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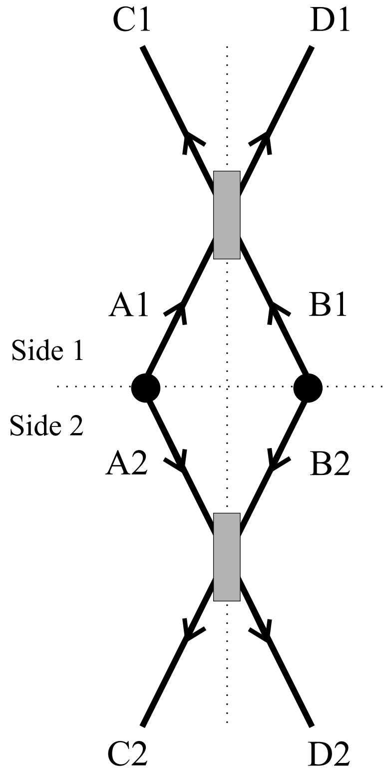

FIG. 1. This figure presents our setup for spin-space entanglement transfer. Each black circle represents a source of a pair of particles maximally entangled in the internal degrees of freedom (not explicitly shown in the figure). The rectangles represent beam splitters.

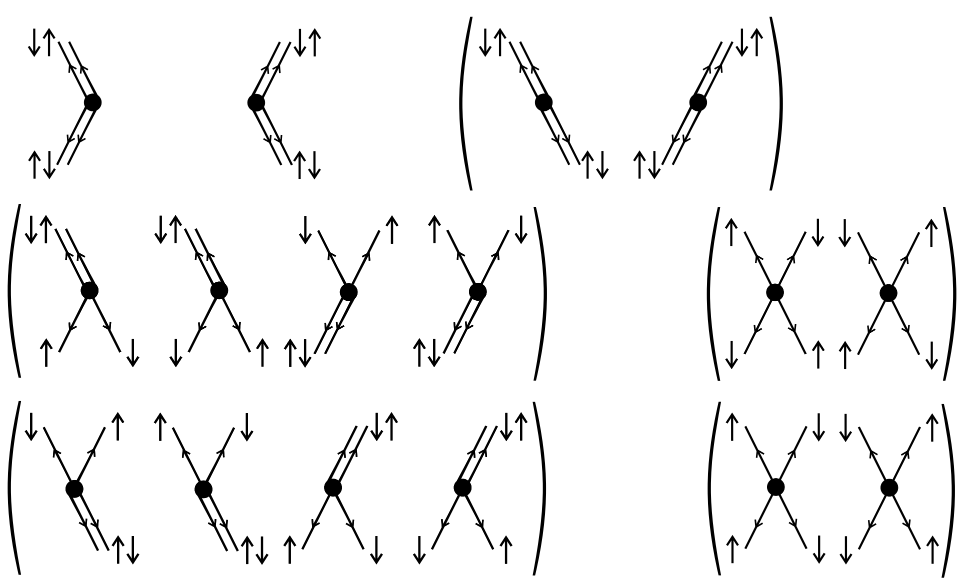

FIG. 2. Spin $S = 0$ component of the total output wave function for the $(+,+)$ case, both for fermions and bosons.

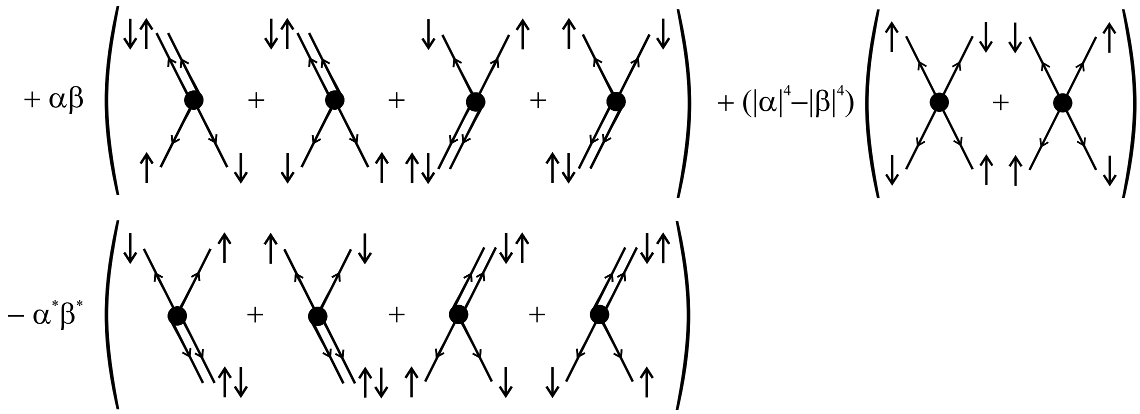

FIG. 3. Spin $S = 0$ component of the total output wave function for the $(+,-)$ case, both for fermions and bosons.

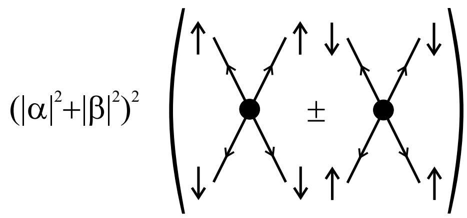

FIG. 4. Spin $S = 1$ component of the total output wave function for the $(+,\pm)$ cases, for fermions.

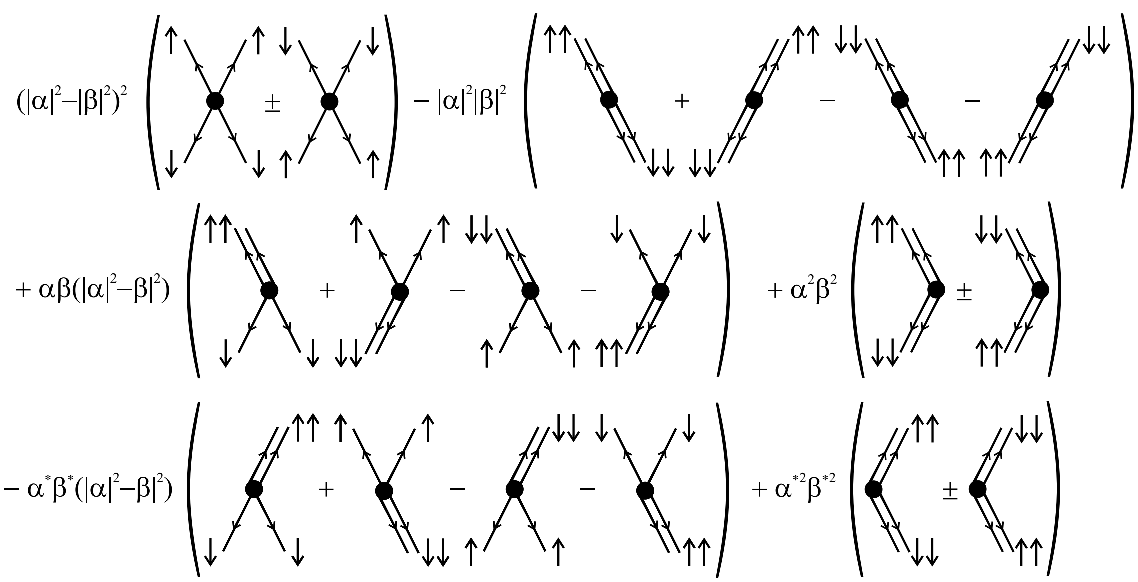

FIG. 5. Spin $S = 1$ component of the total output wave function for the $(+,\pm)$ cases, for bosons.

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